

# Sine and cosine equations on commutative hypergroups

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## Abstract

In this paper we describe the solutions of the functional equations expressing the addition theorems for sine and cosine on commutative hypergroups.

## 1 Introduction

In this paper  $\mathbb{C}$  denotes the set of complex numbers. By a *hypergroup* we mean a locally compact hypergroup. The identity element of the hypergroup  $K$  will be denoted by  $o$ .

For basics about hypergroups see the monograph [4]. The detailed study of functional equations on hypergroups started with the papers [6, 7, 8]. A comprehensive monograph on the subject is [17]. Further results and references on this topic can be found in [9, 10, 11, 12, 13, 15, 16]. Concerning other similar trigonometric-type functional equations the reader should consult with [1, 2, 3, 5, 18].

In this paper we study the *sine-cosine functional equation*

$$f(x * y) = f(x)g(y) + f(y)g(x) \quad (1)$$

and the *cosine-sine functional equation*

$$g(x * y) = g(x)g(y) - f(x)f(y) \quad (2)$$

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on an arbitrary commutative hypergroup  $K$ . In case of both equations we shall always assume that  $f, g : K \rightarrow \mathbb{C}$  are non-identically zero continuous functions.

We note that these functional equations are fundamental in the theory of functional equations. In particular, if in (1) we have  $g = 1$ , then  $f$  is an *additive function*, that is

$$f(x * y) = f(x) + f(y), \quad (3)$$

and if in (2) we have  $h = 0$ , then  $g$  is an *exponential*, that is

$$g(x * y) = g(x)g(y). \quad (4)$$

We shall see that in the case of the sine-cosine equation (1) the situation is very sophisticated if  $g = m$  is an exponential. In the group case exponentials are never zero, hence we can divide by  $m$  and we deduce immediately that  $f$  has the form  $f = a \cdot m$ , where  $a$  is additive. Obviously, this is a solution of (1) on any commutative hypergroup, but there, in general, we cannot divide by  $m$  as exponentials on hypergroups may take zero values. It turns out that on a general commutative hypergroup the solutions  $f$  of (1) with an exponential  $g = m$  produce a new basic function class, which cannot be described directly using exponentials and additive functions. This is a new feature provided by the delicate structure of hypergroups and it seems reasonable to introduce the following definition: if  $K$  is a commutative hypergroup and  $m$  is an exponential on  $K$ , then the function  $f : K \rightarrow \mathbb{C}$  will be called an  *$m$ -sine function*, if it satisfies

$$f(x * y) = f(x)m(y) + f(y)m(x)$$

for each  $x, y$  in  $K$ . We call  $f$  a *sine function*, if it is an  $m$ -sine function for some exponential  $m$ . Obviously, we have  $f(o) = 0$  for every sine function  $f$ . Additive functions on  $K$  are exactly the 1-sine functions. If  $K = G$  is an Abelian group, then for a given exponential  $m$  the  $m$ -sine functions are exactly the functions of the form  $f = a \cdot m$ , where  $a$  is additive. But this is not the case on hypergroups. For instance, let  $K$  be the polynomial hypergroup generated by the sequence of polynomials  $(P_n)_{n \in \mathbb{N}}$  (see [4]). It is known that all exponential functions on  $K$  have the form  $n \mapsto P_n(\lambda)$  with some complex number  $\lambda$ , and all additive functions have the form  $n \mapsto cP'_n(0)$  with some complex number  $c$  (see [17]). Further, if we define

$$f(n) = P'_n(\lambda), \quad m(n) = P_n(\lambda)$$

for each  $n$  in  $\mathbb{N}$  with some complex number  $\lambda$ , then it is easy to check that the following equation holds for each  $m, n$  in  $\mathbb{N}$ :

$$f(n * k) = f(n)m(k) + f(k)m(n),$$

that is,  $f$  is an  $m$ -sine function. On the other hand, it is easy to see that it does not have the form  $n \mapsto cP'_n(0)P_n(\lambda)$  for any complex  $c$ .

In the forthcoming paragraphs we shall describe the solutions of the sine-cosine and the cosine-sine functional equations (1) and (2) on arbitrary commutative hypergroups in terms of exponentials and sine functions.

## 2 Sine-cosine functional equations on hypergroups

In this section we describe the nonzero solutions of the sine-cosine functional equation (1) on commutative hypergroups.

**Theorem 1.** *Let  $K$  be a commutative hypergroup, and let  $f, g : K \rightarrow \mathbb{C}$  be non-identically zero continuous functions satisfying (1) for each  $x, y$  in  $K$ . Then there exists a complex number  $c \neq 0$  and there are continuous exponentials  $M, N : K \rightarrow \mathbb{C}$  such that we have one the following possibilities:*

i)  $g(x) = M(x)$ , and  $f$  is an  $M$ -sine function.

ii)

$$f(x) = \frac{1}{2c} M(x), \quad g(x) = \frac{1}{2} M(x) \quad (5)$$

for each  $x$  in  $K$ .

iii)

$$f(x) = \frac{1}{2c} [M(x) - N(x)], \quad g(x) = \frac{1}{2} [M(x) + N(x)] \quad (6)$$

for each  $x$  in  $K$ .

Conversely, the functions  $f, g$  given above are continuous solutions of (1) for every nonzero complex number  $c$  and continuous complex exponentials  $M, N$ .

*Proof.* As the case i) obviously describes a possible solution, hence we will suppose that  $g$  is not an exponential.

Suppose first that  $g(o) \neq 1$ . By substitution  $y = o$  into (1) we get

$$f(x)(1 - g(o)) = f(o)g(x),$$

that is  $f(x) = \frac{1}{c}g(x)$  with some complex number  $c \neq 0$ . It follows from (1)

$$2g(x * y) = 2g(x)2g(y)$$

hence  $g = \frac{1}{2}m$  and  $f = \frac{1}{2c}m$ , where  $m$  is an exponential, which is given in ii) with  $M = m$ .

Now we assume  $g(o) = 1$ . By substitution  $y = o$  into (1) we get

$$f(x) = f(x)g(o) + f(o)g(x) = f(x) + f(o)g(x)$$

which implies  $f(o)g(x) = 0$ , hence  $f(o) = 0$ .

We introduce the Cauchy difference: for each  $x, y$  in  $K$  we define

$$F(x, y) = f(x * y) - f(x) - f(y)$$

which can be written as

$$F(x, y) = f(x)[g(y) - 1] + f(y)[g(x) - 1].$$

Obviously,  $F$  satisfies

$$F(x, y) + F(x * y, z) = F(x, y * z) + F(y, z)$$

for each  $x, y, z$  in  $K$ , by the associativity of the hypergroup operation. After substitution and simplification we get the equation

$$f(z)[g(x * y) - g(x)g(y)] = f(x)[g(y * z) - g(y)g(z)].$$

for each  $x, y, z$  in  $K$ . As  $f \neq 0$ , we have

$$g(x * y) = g(x)g(y) + f(x)\varphi(y)$$

with some continuous  $\varphi : K \rightarrow \mathbb{C}$ . By commutativity, we obtain  $\varphi = \lambda f$  with some complex number  $\lambda$ . We write  $\lambda = -d^2$  and we infer

$$g(x * y) = g(x)g(y) - df(x)df(y),$$

or, with the notation  $h = df$  we obtain (2) for  $g$  and  $h$ . Here  $d \neq 0$ , as otherwise  $g$  is an exponential and we have  $i$ ).

Then we multiply (1) by  $d$  and we have the system for the pair  $g, h$ :

$$h(x * y) = h(x)g(y) + h(y)g(x) \tag{7}$$

$$g(x * y) = g(x)g(y) - h(x)h(y)$$

for each  $x, y$  in  $K$ . Let for each  $x$  in  $K$ :

$$M(x) = g(x) + ih(x), \quad \text{and} \quad N(x) = g(x) - ih(x).$$

We have for each  $x, y$  in  $K$

$$M(x * y) = g(x * y) + ih(x * y) = g(x)g(y) - h(x)h(y) + ig(x)h(y) + ig(y)h(x) =$$

$$(g(x) + ih(x))(g(y) + ih(y)) = M(x)M(y),$$

and

$$N(x * y) = g(x * y) - ih(x * y) = g(x)g(y) - h(x)h(y) - ig(x)h(y) - ig(y)h(x) =$$

$$(g(x) - ih(x))(g(y) - ih(y)) = N(x)N(y).$$

This means that  $M, N : K \rightarrow \mathbb{C}$  are exponentials. On the other hand, we have

$$g = \frac{1}{2}(M + N), \quad h = \frac{1}{2i}(M - N).$$

It follows  $f = \frac{1}{2di}(M - N)$ , and we have  $iii$ ) with  $c = di$ .

The converse statement can be verified easily by direct computation. □

### 3 Cosine-sine functional equations on hypergroups

In this section we describe the nonzero solutions of the cosine-sine functional equation (2) on commutative hypergroups.

**Theorem 2.** *Let  $K$  be a commutative hypergroup, and let  $f, g : K \rightarrow \mathbb{C}$  be non-identically zero continuous functions satisfying (2) for each  $x, y$  in  $K$ . Then there exist complex numbers  $c \neq 0, 1$  and  $d \neq 0$ , and there are continuous exponentials  $M, N : K \rightarrow \mathbb{C}$  such that we have one of the following possibilities:*

i)

$$f(x) = \frac{c}{1-c^2} M(x), \quad g(x) = \frac{1}{1-c^2} M(x) \quad (8)$$

for each  $x$  in  $K$ .

ii)

$$f(x) = \frac{1}{2c} M(x), \quad g(x) = \frac{1}{2} M(x) \quad (9)$$

for each  $x$  in  $K$ .

iii)  $f$  is an  $M$ -sine function, and  $g(x) = M(x) \pm f(x)$  for each  $x$  in  $K$ .

iv)

$$f(x) = \pm \frac{1}{2di} [M(x) - N(x)], \quad g(x) = \pm \frac{\pm di - \lambda}{2di} M(x) \pm \frac{\pm di + \lambda}{2di} N(x) \quad (10)$$

for each  $x$  in  $K$ , where  $d^2 = 1 - \lambda^2$ , and we choose  $+$  or  $-$  at each place in the same way.

Conversely, the functions  $f, g$  given above are continuous solutions of equation (2) for any nonzero complex numbers  $c, d$ ,  $c \neq \pm 1$  and continuous complex exponentials  $M, N$ .

*Proof.* First we note that here  $g$  is not an exponential, otherwise  $f$  is identically zero. Substituting  $y = o$  we have  $g(x)(1 - g(o)) = -f(x)f(o)$ , hence if  $g(o) \neq 1$ , then  $g(x) = \frac{1}{c}f(x)$  with some complex number  $c \neq 0$ . We also have  $c \neq \pm 1$  otherwise  $g = \pm f$  and substitution into (2) gives that  $f = g = 0$ . It follows from (2)

$$\frac{1-c^2}{c} f(x * y) = \frac{1-c^2}{c} f(x) \frac{1-c^2}{c} f(y)$$

implying  $f(x) = \frac{c}{1-c^2} m(x)$  and  $g(x) = \frac{1}{1-c^2} m(x)$  with some exponential  $m$ , which is i) with  $M = m$ .

Now we assume  $g(o) = 1$ , and in this case (2) implies  $f(o) = 0$ . We define a modified Cauchy difference  $G(x, y) = g(x * y) - g(x)g(y)$  such that we have

$$g(z)G(x, y) + G(x * y, z) = G(x, y * z) + g(x)G(y, z)$$

for each  $x, y, z$  in  $K$ . Then, by (2), it follows

$$f(x)[f(y * z) - f(y)g(z)] = f(z)[f(x * y) - g(x)f(y)].$$

As  $f \neq 0$  this implies

$$f(x * y) = f(x)\varphi(y) + f(y)g(x)$$

with some continuous  $\varphi : K \rightarrow \mathbb{C}$ . Interchanging  $x$  and  $y$  we have the relation

$$\varphi(x) = g(x) + 2\lambda f(x)$$

with some complex number  $\lambda$ . If  $\lambda = 0$ , then we have  $\varphi = g$ , and the pair  $f, g$  satisfies the sine equation (1). Case *i*) in Theorem 1 cannot occur. Case *ii*) in Theorem 1 gives  $c = \pm i$ , which is included in *i*) above with  $c = \pm i$ . Finally, case *iii*) in Theorem 1 gives  $c = \pm i$  which is included in case *iii*) above with  $c = \pm i$ .

Now we assume  $\lambda \neq 0$ , then we have

$$f(x * y) = f(x)g(y) + 2\lambda f(x)f(y) + f(y)g(x). \quad (11)$$

We introduce the function

$$h(x) = g(x) + \lambda f(x),$$

then a simple calculation shows that

$$f(x * y) = f(x)h(y) + f(y)h(x) \quad (12)$$

and

$$h(x * y) = h(x)h(y) - (1 - \lambda^2)f(x)f(y). \quad (13)$$

Equation (12) shows that  $f$  and  $h$  satisfy the sine-cosine equation (1), hence we have the description of the solutions, we just have to extract the solutions of (2). But we also have to consider equation (13) which depends on  $\lambda$ . If  $\lambda^2 = 1$ , then  $h = m$  is an exponential and  $f$  is an  $m$ -sine function. In this case we have  $g = m \pm f$  and substitution into (2) gives that this is a solution indeed, which is covered by case *iii*) above.

Finally we suppose  $\lambda^2 \neq 1$ . We take  $d \neq 0$  with  $d^2 = 1 - \lambda^2$ , then we have by (13)

$$h(x * y) = h(x)h(y) - df(x)df(y)$$

and, multiplying (12) by  $d$  gives

$$df(x * y) = df(x)h(y) + df(y)h(x)$$

for each  $x, y$  in  $K$ . This means that the pair  $df, h$  satisfies the sine-cosine and the cosine-sine functional equations simultaneously, and in this case  $h$  is not an

exponential, hence we have to consider cases *ii*) and *iii*) only, in Theorem 1. In case *ii*) we get  $c = \pm i$  and, by the definition of  $h$

$$f(x) = \pm \frac{1}{2di} M(x), \quad g(x) + \lambda f(x) = \frac{1}{2} M(x)$$

which implies that  $f$  and  $g$  are constant multiples of each other, hence we have case *i*) above. In case *iii*) of Theorem 1 we obtain

$$f(x) = \frac{1}{2cd} [M(x) - N(x)], \quad g(x) = \frac{cd - \lambda}{2cd} M(x) + \frac{cd + \lambda}{2cd} N(x).$$

Substitution into (2) gives  $c = \pm i$  and we have

$$f(x) = \pm \frac{1}{2di} [M(x) - N(x)], \quad g(x) = \pm \frac{\pm di - \lambda}{2di} M(x) \pm \frac{\pm di + \lambda}{2di} N(x),$$

where  $d^2 = 1 - \lambda^2$ , and we choose  $+$  or  $-$  at each place in the same way, as it is given in case *iv*) above.

The converse statement can be verified easily by direct computation.  $\square$

Using results concerning the form of exponentials on some particular hypergroups discussed in [17] one can obtain explicit forms of sine functions on certain hypergroups.

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